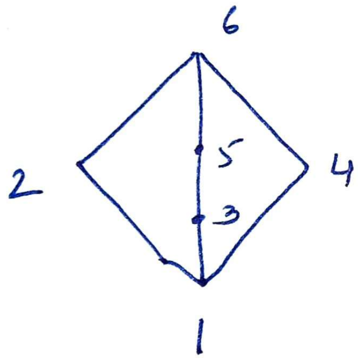
8. **Atom:** Let $(L, \vee, \wedge)$ be Lattice under $\leq$ with l.b 0. An Element a is called an atom if it is an immediate successor of 0.

i.e. $a \neq 0$ is called atom if $0 \leq b \leq a$ then $b = 0$ or $b = a$.

e.g. 1.



l.b = 1

Atoms = 2, 3, 4

$\because$ [2, 3 & 4 is successor of 1].

Every Boolean Algebra has no. of Elements $= 2^n$

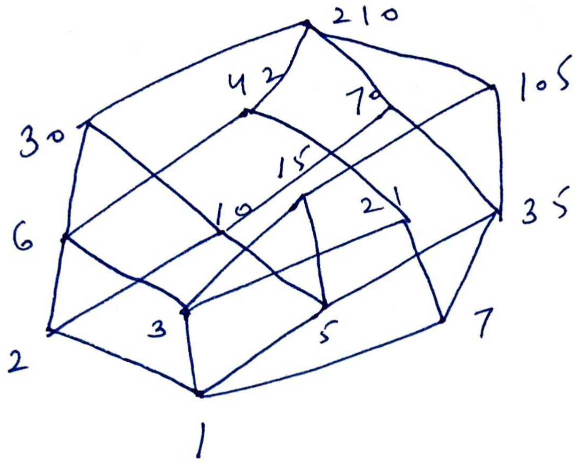
n = No. of atoms

Q2: Is D_{210} is a Boolean algebra? List its Elements & find its Atoms and find all Sub Algebras.

Sol: $210 = 2 \cdot 3 \cdot 5 \cdot 7$ (Product of Primes)

$\therefore$ 210 is a Boolean Algebra.

$D_{210} = \{1, 2, 3, 5, 7, 6, 10, 14, 15, 30, 35, 42, 21, 70, 105, 210\}$



least Element = 1

Atom = $\{2, 3, 5, 7\}$ ($2^4 = 16$ Elements)

Any Subalgebra of D_{210} have 2, 4, 8, 16 Elements

i) Sub algebra with 2 Elements shall Contain least Elements, Greatest Element.

$\therefore B_1 = \{1, 210\}$

ii) Sub Algebra with 4 Elements shall Contain l. Ele., great. Ele., an atom and its Complement.

$\therefore B_2 = \{1, 2, 105, 210\}$

$B_3 = \{1, 3, 70, 210\}$

$B_4 = \{1, 5, 42, 210\}$

$B_5 = \{1, 7, 30, 210\}$

[Comlement = $\frac{n}{\text{atom (Element)}}$]

iii) Sub algebra with 8 Elements shall contain l.E, g.E, two atoms, Product of atoms and there complements.

$B_6 = \{1, 2, 3, 6, 105, 70, 35, 210\}$

$B_7 = \{1, 2, 5, 10, 105, 42, 21, 210\}$

$B_8 = \{1, 2, 7, 14, 105, 30, 15, 210\}$

$B_9 = \{1, 3, 5, 15, 70, 42, 14, 210\}$

$B_{10} = \{1, 3, 7, 21, 70, 30, 10, 210\}$

$B_{11} = \{1, 5, 7, 35, 42, 30, 6, 210\}$

iv) Sub algebra of 16 Elements is D_{210} itself.

OR

ii) $\{l.b, x, x', g.b\}$

Non-bound Elements in $D_{210} = \frac{14}{2} = 7$ Pairs

$\left(x' = \frac{210}{\text{Element}}\right)$

$\therefore$ $B_2 = \{1, 2, 105, 210\}$

$B_3 = \{1, 3, 70, 210\}$

$B_4 = \{1, 5, 42, 210\}$

$B_5 = \{1, 6, 35, 210\}$

$B_6 = \{1, 7, 30, 210\}$

$B_7 = \{1, 10, 21, 210\}$

$B_8 = \{1, 14, 15, 210\}$

Hence No. of Sub Algebra of 210 $= 1 + 7 + 6 + 1$

$= \underline{15}$.

Thm: Let B finite Boolean algebra. Then every non-zero element in B is join of unique set of atoms.

Proof: Let $x \in B$, $\{a_1, a_2 \dots a_n\}$ - set of atoms of B.

STEP I: $a_1, a_2 \dots a_n$ are atoms of B

$\therefore a_1 \leq x, a_2 \leq x, \dots a_n \leq x$

$\Rightarrow a_1 \vee a_2 \vee \dots \vee a_n \leq x$ — (1)

Let $a_1 \vee a_2 \vee \dots \vee a_n = y$

$\therefore y \leq x$ — (2)

T.P $x \wedge y' = 0$

If possible let $x \wedge y' > 0$

$\exists$ atom

$0 < a \leq 0$

$0 < a \leq x \wedge y'$

$a \leq x$ & $a \leq y'$

Now $a \leq x$

$a = a_i$

$a \leq a_1 \vee a_2 \vee \dots \vee a_n$

$a \leq y$ — (3)

Also we have $a \leq y'$

$\therefore a \leq y \wedge y'$

$a \leq 0$ Not possible

$\therefore x \wedge y' = 0$.

Step II $x = x \wedge 1$

$= x \wedge (y \vee y')$

$= (x \wedge y) \vee (x \wedge y')$

$= (x \wedge y) \vee 0$

$x = x \wedge y$

$\therefore x \leq y$ — (3)

By (2) & (3)

$x = y$

$x = a_1 \vee a_2 \vee \cdots \vee a_n$

Uniqueness : Let $x = b_1 \vee b_2 \vee \cdots \vee b_m$

$b_1, b_2 \cdots b_m \to$ atoms

$\therefore b_j \leq x$

$\Rightarrow b_j \wedge x = b_j$

$\Rightarrow b_j \wedge (a_1 \vee a_2 \cdots \vee a_n) = b_j$

$(b_j \wedge a_1) \vee (b_j \wedge a_2) \cdots \vee (b_j \wedge a_n) = b_j > 0$

$\therefore b_j \wedge a_i \neq 0$

$a_i, b_j \in B$

a_i is atom $a_i \leq b_j$

lly $b_j \leq a_i$

$\Rightarrow a_i = b_j$

Hence Proved.